The noise spectrum in the model of self-organised criticality

1990 J. Phys. A: Math. Gen. 23 L433

(http://iopscience.iop.org/0305-4470/23/9/006)

J. Phys. A: Math. Gen. **23** (1990) L433–L440. Printed in the UK

LETTER TO THE EDITOR

The noise spectrum in the model of self-organised criticality*

János Kertész† and László B Kiss‡

Institute for Theoretical Physics, University of Cologne, Zülpicher Strasse 77, D-5000 Köln 41, Federal Republic of Germany

Received 15 January 1990

Abstract. The explanation of $1/f$ noise with the help of models of self-organised criticality makes use of combining scaling and the distributed time-constant picture. Limits of applicability of this type of arguments are presented. The numerical study of the two-dimensional model of Bak, Tang and Wiesenfeld shows that although non-trivial scaling is present, the power spectrum of the total process has a $\sim 1/f^2$ behaviour.

Bak, Tang and Wiesenfeld (BTW) (1987, 1988) introduced a very interesting model to explain the fractality emerging spontaneously in nature as well as the so-called $1/f$ flicker noise. The basic concept is that the considered non-equilibrium systems build up 'self-organised criticality' (SOC), i.e. a stationary state without finite characteristic length and time scales (besides the microscopic units) which leads to power law behaviour in the characteristic quantities like the cluster size distribution, the distribution of the lifetimes of single events, etc. In order to visualise the ideas, BTW used the terminology of sandpiles (single events) and avalanches (total system) but they emphasised that the applicability of the concepts is much more general.

BTW stressed that SOC should be distinguished from usual critical phenomena where the lack of scales appears only at a special value of the parameters which should be forced on to the system to bring it to criticality. Although there are parameters also in the BTW model which should be kept at their critical value (e.g. the limit small energy supply is considered), it seems that the requirements can be fulfilled to considerable extent leading to fractal behaviour over many orders of magnitudes.

One possible important application of the concept of SOC is in the area of $1/f$ noise. In spite of the universal occurrence of $1/f$ (or $1/f^{\alpha}$) noise and the large effort investigators have made, a general theory of this phenomenon does not exist (Kiss 1988). It has been an appealing idea for some time to trace back the power law behaviour of the noise spectrum to the scale invariance of fractal or critical systems. BTW suggested that the structures emerging from SOC and having scaling properties both in space and time lead to a $1/f$-type noise spectrum. The arguments were based on a combination of scaling ideas with the established formula describing the noise of Lorentzian fluctuators with distributed time constants.

* The numerical results of this paper were briefly reported at the 10th Int. Conf. on Noise in Physical Systems, Budapest, August 21–5, 1989.
† On leave from Institute for Technical Physics, H-1325 Budapest, Hungary.
‡ On leave from Institute for Experimental Physics, JATE University, Dómtér 9, Szeged, H-6720 Hungary.

0305-4470/90/090433+08$03.50

Recently some experiments were carried out where the noise spectrum was measured in systems which could be related to SOC. Jaeger *et al* (1989) found a $1/f^2$ spectrum by investigating the flow of sand. A similar spectrum was found by Janossi and Horváth (1989), who measured the coverage fluctuations of water droplets on a window pane.

The purpose of this letter is to study the following questions. (i) What are the theoretical limits of the application of scaling arguments to obtain $1/f^\alpha$-type noise with a non-trivial α? (ii) What is the numerically determined noise spectrum in the BTW model?

Let us suppose that in a non-equilibrium system independent elementary events contribute to the energy dissipation. We shall use here the language of BTW, i.e. the elementary events are identified with the avalanches, but the considerations are general. An elementary event is characterised by the time dependence of its amplitude $\tilde{f}_s(t)$ where s is the total impact (number of moves during an avalanche). Introducing the Fourier transform $f_s(\omega) = \int \exp(\mathrm{i}\omega t)\tilde{f}_s(t)\,\mathrm{d}t$, the mean energy density spectrum of avalanches with a given size s is defined as

$$\langle E_s(\omega)\rangle = \langle |f_s(\omega)|^2\rangle \tag{1}$$

where $\langle \ldots \rangle$ means statistical average.

Under quite general conditions the energy density spectrum† of non-oscillating elementary events of a given size can be considered as a (quasi-)Lorentzian. (Nevertheless, this is an assumption which has to be checked in specific situations.) Using the normalisation condition for $f(\omega = 0) = s$ one gets

$$\langle E_s(\omega)\rangle = \frac{s^2}{1+\omega^2 T_s^2} \tag{2}$$

where T_s is the mean time constant of the avalanches of size s. Since the avalanches do not interact, the total power density spectrum $S(\omega)$ is the weighted sum of the individual contributions:

$$S(\omega) \sim \int n_s \frac{s^2\,\mathrm{d}s}{1+\omega^2 T_s^2} \tag{3}$$

where n_s is the density distribution of avalanches of size s.

In a scaling framework T_s and n_s should have power law behaviour:

$$T_s \sim s^x, \tag{4a}$$

$$n_s \sim s^{-\tau}. \tag{4b}$$

Substituting these relations into (3) we get:

$$S(\omega) \sim \int_{s_1}^{s_2} \frac{s^{2-\tau}}{1+\omega^2 s^{2x}}\,\mathrm{d}s \sim \frac{1}{\omega^{(3-\tau)/x}} \int_{\omega T_{s_1}}^{\omega T_{s_2}} \frac{u^{(3-x-\tau)/x}}{1+u^2}\,\mathrm{d}u. \tag{5}$$

Here s_1 (s_2) and T_{s_1} (T_{s_2}) are the sizes and lifetimes at the lower (upper) cut-off. Since we are interested in the low-frequency behaviour it is only the upper cut-off (long-time behaviour) which has to be considered. Clearly, for

$$2x + \tau > 3 \tag{6}$$

† The name energy density spectrum is somewhat misleading and has historical origins (voltage fluctuations). In the case of SOC the amplitude of the elementary event (avalanche) is proportional to the dissipated energy; therefore the noise spectrum goes with the square of this quantity.

the integral is convergent as $T_{s_2} \to \infty$ and

$$S(\omega) \sim \frac{1}{\omega^{(3-\tau)/x}} \tag{7}$$

follows. In particular, for $x+\tau=3$ we get the exact $1/f$ behaviour.

On the other hand, for $2x+\tau \leqslant 3$ the upper cut-off contributes to the frequency dependence since the weight of the $T_s \gg 1/\omega$ modes becomes important:

$$S(\omega) \sim (\omega T_{s_2})^{(3-2x-\tau)/x} \frac{1}{\omega^{(3-\tau)/x}} = T_{s_2}^{(3-2x-\tau)/x} \frac{1}{\omega^2}. \tag{8}$$

In this case the low-frequency behaviour will always be $1/\omega^2$ (with logarithmic correction for $2x+\tau=3$) and the amplitude will increase with the upper cut-off T_{s_2}. A similar result using special assumptions was obtained by Jensen *et al* (1989).

The essential points going into these considerations have been: (a) the contributions to the total energy dissipation stem from independent elementary events, (b) the asymptotic shape of the mean energy spectrum of a single event is $\langle E_s \rangle \sim 1/(\omega T_s)^2$ (cf equation (2)), (c) T_s and n_s obey scaling forms according to (4). Changing assumption (b) to

$$\lim_{s \to \infty} \langle E_s \rangle \sim s^2/(\omega T_s)^{2\delta} \tag{9}$$

we would get

$$2x\delta + \tau > 3 \tag{10}$$

instead of criterion (6) while, in case (10) is violated, we get

$$S(\omega) \sim T_{s_2}^{(3-2x\delta-\tau)/x} \frac{1}{\omega^{2\delta}}. \tag{11}$$

It is sometimes useful to express the above relations in terms of the lifetime distribution n_T which can also be assumed to obey a power law:

$$n_T \sim T^{-y}. \tag{12a}$$

The relation $n_T\, \mathrm{d}T \sim n_T(\mathrm{d}T/\mathrm{d}s)\, \mathrm{d}s \sim n_s\, \mathrm{d}s$ and (4) lead to the following scaling law:

$$x(1-y) = 1-\tau. \tag{12b}$$

With this result the scaling limit of the power spectrum takes the form

$$S(\omega) \sim \frac{1}{\omega^{(3-\tau)(1-y)/(1-\tau)}}. \tag{13}$$

So far our considerations are general and not system or model dependent. Now we turn to the concrete model of BTW.

We repeated the computer modelling of the original two-dimensional self-organised critical system proposed by BTW; however, we determined all the density functions considered in the previous sections, including the energy density spectra and the power density spectrum of the total avalanche current. A square array (x, y) was taken with $0 < x, y < N+1$ ($N = 10, 20, 40, 80$).

The model is defined in the following way (Bak *et al* 1988). As a consequence of an excitation at x, y the neighbourhood is changed:

$$z(x,y) \to z(x,y)+2 \tag{14a}$$

$$z(x-1,y) \to z(x-1,y)-1 \tag{14b}$$

$$z(x,y-1) \to z(x,y-1)-1. \tag{14c}$$

If the local variable $z(x, y)$ exceeds a given critical parameter z_c a 'slide' takes place to the neighbouring points:

$$z(x,y) \to z(x,y)-4 \tag{15a}$$

$$z(x,y\pm 1) \to z(x,y\pm 1)+1 \tag{15b}$$

$$z(x\pm 1,y) \to z(x\pm 1,y)+1. \tag{15c}$$

We carried out simulations in the weak-excitation limit, i.e. we started the avalanches by a single excitation and then observed the possible spreading sequence by slidings determined by this event. Before the measurement of the statistical properties of the avalanches, the system was run (in the way described above) many times, until a stationary state (the SOC state) was reached.

Here we present the numerical results found on the system of size $N=80$. In the critical range, the weighted energy density spectra $n_s E_s(\omega)$ of the avalanches turned out to scale with the size s (see figure 1):

$$n_s E_s(\omega) \sim s^z g(\omega s^x) \tag{16}$$

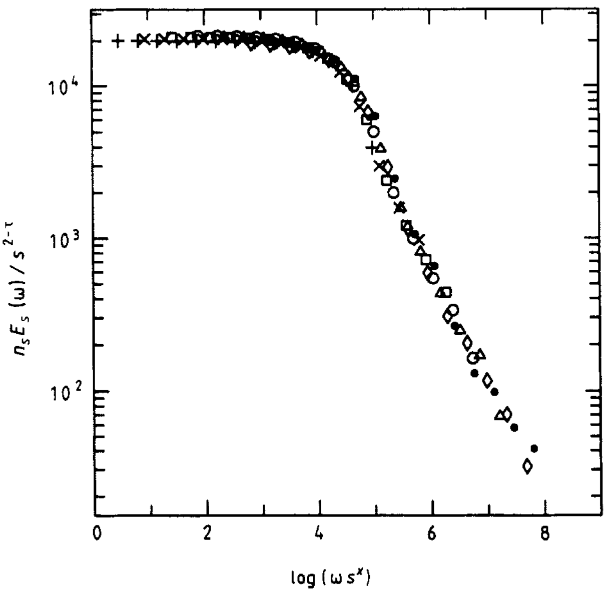

Figure 1. The scaling of elementary (avalanche) spectra. The exponent z in the text is equal to $2-\tau=0.9$. Symbols (avalanche sizes): cross (4); square (8); circle (16); triangle (32); diamond (64); dot (128).

with $z=0.9$ and $x=0.68$, with g being a scaling function which can be well approximated by a Lorentzian. That means it is nearly constant at low frequencies and nearly $1/\omega^2$ above a certain corner frequency $1/T_s$ (see (2)):

$$E_s(\omega)\approx\frac{s^2}{1+\omega^2T_s^2}. \tag{17}$$

Comparing (16) and (17) we get for the size distribution in the critical range that

$$n_s\sim s^{-1.1} \tag{18}$$

that is, $\tau=1.1$ (see 4(b)) and the dependence of the mean time constants on the size of the avalanches should be:

$$T_s\sim s^{0.68} \tag{19}$$

that is, $x=0.68$ (see 4(a)). Both these relations (18) and (19) are confirmed by our direct measurements; see figures 2 and 3.

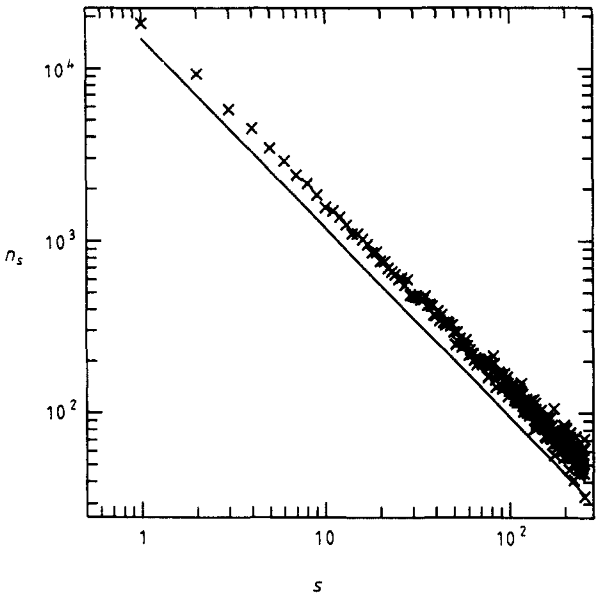

Figure 2. Avalanche size distribution. The full line follows a power law with exponent $-\tau=-1.1$.

Using (12b) we can predict that the distribution of time constants has the following form:

$$n_T\sim T^{-1.115} \tag{20}$$

that is, $y=1.115$. As can be seen on figure 4, a $T^{-1.115}$ fit of the measured n_T is a fairly good approximation†.

Taking the above values we see that condition (6) is violated:

$$2x+\tau=2.46<3. \tag{21}$$

† The exponents characterising the distributions (τ and y) are in agreement with BTW within numerical accuracy. (Note that our τ is related to the τ_{BTW} as $\tau+1=\tau_{\mathrm{BTW}}$.) Recently, by careful large-scale computations, Manna (1989) found more accurate and somewhat different exponents.

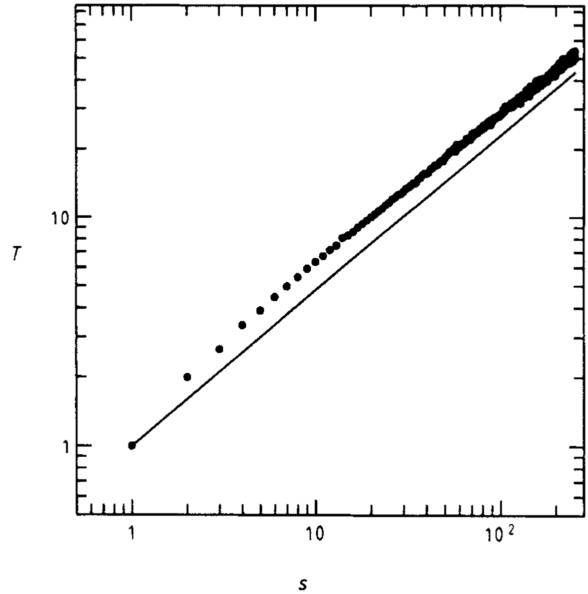

Figure 3. Avalanche lifetime against size. The full line follows a power law with exponent $x = 0.68$.

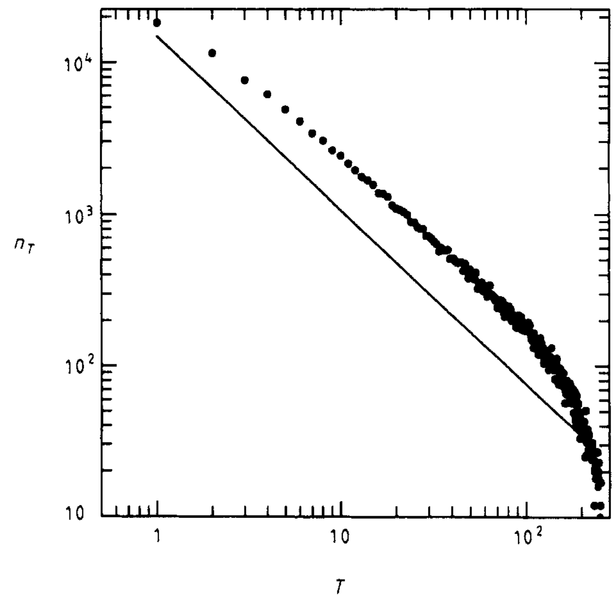

Figure 4. Distribution of avalanche lifetimes. The full line represents a power law with exponent $-y = -1.115$.

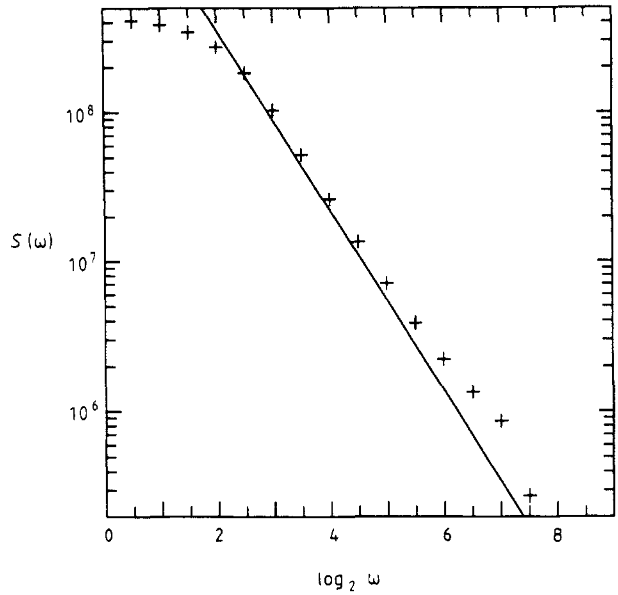

Figure 5. The total power density spectrum of the system of avalanches. It is a Lorentzian-like spectrum, dominated by the longest avalanches. The full line represents a $1/\omega^2$ spectrum.

Consequently, one can predict a $1/f^2$-like power density spectrum of the total process. In figure 5, the total power spectrum can be seen which is indeed $1/f^2$-like in the critical frequency range (that is, for frequencies higher than the reciprocal cut-off time given by the system size 80). This spectrum is dominated by the spectra of the avalanches with the longest duration.

Our results can be summarised as follows. We have shown that the scaling exponents determine the exponent of the noise spectrum only if the exponent inequalities (6) or (10) are valid. The obtained relation (7) is different from that suggested earlier by Tang and Bak (1988) who did not use the proper weight in the noise spectrum. Jensen *et al* (1989) realised this discrepancy but made some unjustified assumptions. Our simulations show that for the two-dimensional model by Bak *et al* the single avalanche noise spectrum is quasi-Lorentzian and the actual values of the exponents violate (6). Therefore we obtained $1/\omega^2$ dependence for the total power spectrum with a size-dependent amplitude in accordance with the fact that the contribution stems from the cut-off and is also in agreement with the recent independent simulation by Jensen *et al.* Although the mechanism suggested for the $1/f$ noise by Bak *et al* is very appealing, it seems that it is not as general as originally believed.

We wish to thank Dietrich Wolf for valuable discussions. The support by the Humboldt Foundation (JK) and by SFB 341 (LBK) as well as by OTKA is gratefully acknowledged.

References

Bak P, Tang C and Wiesenfeld K 1987 *Phys. Rev. Lett.* **59** 381
—— 1988 *Phys. Rev.* A **38** 364

Jaeger H M, Liu Chu-heng and Nagel S R 1989 *Phys. Rev. Lett.* **62** 40
Jánosi I M and Horváth V K 1989 *Phys. Rev.* A **40** 5232
Jensen J H, Christensen K and Fogedby H C 1989 *Phys. Rev.* B **40** 7425
Kiss L B 1988 *Rev. Solid State Sci.* **2** 659
Manna S S 1989 *Physica* to be published
Tang C and Bak P 1988 *Phys. Rev. Lett.* **60** 2347